

Given two vertically adjacent parcels in a fluid, thoroughly mixing the parcels will conserve momentum, but reduce the kinetic energy. If the released kinetic energy is enough energy to raise up the heavier particles (the total potential energy of the mixture minus that of the original distribution), then the parcels will mix via turbulence.

$R \text{ (J/kg*K)}$ = gas constant (287 for dry air)
 $S \text{ (J/kg*K)}$ = specific heat capacity (1003.5 for dry air)
 K = exponent of exner function = R/S
 $M_L \text{ (kg/m}^2)$ = mass per unit area of each layer
 $P_L \text{ (Pa)}$ = mean pressure of each layer
 $V_L \text{ (m/s)}$ = horizontal velocity of each layer
 $T_L \text{ (}^\circ\text{K)}$ = temperature of each layer
 $\theta_L \text{ (}^\circ\text{K)}$ = mean potential temperature of each layer = $T_L * (P_0/P_L)^K$
 $\alpha_L \text{ (m}^2/\text{kg)}$ = specific volume of each layer = $R*T_L/P_L$
 $\beta_L \text{ (m}^2/\text{kg)}$ = potential specific volume of each layer = $R*\theta_L/P_0$
 $s\theta_L \text{ (J/kg)}$ = potential specific enthalpy of each layer = $S*\theta_L$
 $P_0 \text{ (Pa)}$ = reference pressure for potential temperature or
 potential specific volume

 $V_A \text{ (m/s)}$ = average velocity of both layers = $(M_1*V_1+M_2*V_2) / (M_1+M_2)$
 $s\theta_A \text{ (}^\circ\text{K)}$ = average potential specific enthalpy of both layers =
 = $(M_1*s\theta_1+M_2*s\theta_2) / (M_1+M_2)$

 $\Delta KE = .5*(M_1+M_2)*V_A^2 - .5*M_1*V_1^2 - .5*M_2*V_2^2 =$
 $= .5*(M_1+M_2)*(M_1*V_1+M_2*V_2)^2 / (M_1+M_2 - .5*M_1*V_1^2 - .5*M_2*V_2^2) =$
 $= .5*[(M_1*V_1+M_2*V_2)^2 - (M_1+M_2)*M_1*V_1^2 - (M_1+M_2)*M_2*V_2^2] / (M_1+M_2) =$
 $= .5*[2*M_1*V_1*M_2*V_2 - M_2*M_1*V_1^2 - M_1*M_2*V_2^2] / (M_1+M_2) =$
 $= - .5*M_1*M_2*(V_2-V_1)^2 / (M_1+M_2)$

$$P_L^K = [PA + (P_L - PA)]^K \approx PA^K + K * PA^{K-1} * (P_K - PA) + .5 * K * (K-1) * PA^{K-2} * (P_K - PA)^2$$

$$\begin{aligned}\Delta TPE &= S\theta A * [M_1 * (P_1/P_0)^K + M_2 * (P_2/P_0)^K] - S\theta_1 * M_1 * (P_1/P_0)^K - S\theta_2 * M_2 * (P_2/P_0)^K = \\ &= [(M_1 * S\theta_1 + M_2 * S\theta_2) * (M_1 * P_1^K + M_2 * P_2^K) - \\ &\quad - (M_1 + M_2) * S\theta_1 * M_1 * P_1^K - (M_1 + M_2) * S\theta_2 * M_2 * P_2^K] / (M_1 + M_2) * P_0^K = \\ &= (M_1 * S\theta_1 * M_2 * P_2^K + M_2 * S\theta_2 * M_1 * P_1^K - M_2 * S\theta_1 * M_1 * P_1^K - M_1 * S\theta_2 * M_2 * P_2^K) / (M_1 + M_2) * P_0^K = \\ &= M_1 * M_2 * (S\theta_1 * P_2^K + S\theta_2 * P_1^K - S\theta_1 * P_1^K - S\theta_2 * P_2^K) / (M_1 + M_2) * P_0^K = \\ &= - M_1 * M_2 * (S\theta_2 - S\theta_1) * (P_2^K - P_1^K) / (M_1 + M_2) * P_0^K\end{aligned}$$

$$\begin{aligned}\Delta TPE/\Delta KE &= [- M_1 * M_2 * (S\theta_2 - S\theta_1) * (P_2^K - P_1^K) / (M_1 + M_2) * P_0^K] / \\ &\quad / [- .5 * M_1 * M_2 * (V_2 - V_1)^2 / (M_1 + M_2)] = \\ &= 2 * (S\theta_2 - S\theta_1) * (P_2^K / P_0^K - P_1^K / P_0^K) / (V_2 - V_1)^2\end{aligned}$$

Equation of State: $P = R * T / \alpha$

Hydrostatic Assumption: $dP/dZ = -g/\alpha$ or $g * dZ = -\alpha * dP$

$$\alpha/\beta = (R * T / P) / (R * \theta / P_0) = T * P_0 / P * \theta = T * P_0 / P * T * (P_0 / P)^K = (P / P_0)^{K-1}$$

$$\begin{aligned}Ri &= \text{Richardson Number} = \Delta\beta * g * \Delta Z / \beta * \Delta V^2 = -\Delta\beta * \alpha A * \Delta P / \beta A * \Delta V^2 = \\ &= -\Delta\beta * \Delta P * (PA / P_0)^{K-1} / \Delta V^2\end{aligned}$$

αA = average α of both layers

βA = average β of both layers

$PA = .5 * (P_1 + P_2)$

$$P_1^K = [PA + (P_1 - PA)]^K \approx PA^K + K * PA^{K-1} * (P_1 - PA) + .5 * K * (K-1) * PA^{K-2} * (P_1 - PA)^2$$

$$P_2^K = [PA + (P_2 - PA)]^K \approx PA^K + K * PA^{K-1} * (P_2 - PA) + .5 * K * (K-1) * PA^{K-2} * (P_2 - PA)^2$$

$$P_2^K - P_1^K \approx K * PA^{K-1} * (P_2 - P_1)$$

$$\begin{aligned}.5 * \Delta TPE/\Delta KE &= (S\theta_2 - S\theta_1) * (P_2^K / P_0^K - P_1^K / P_0^K) / (V_2 - V_1)^2 = \\ &= K * (S\theta_2 - S\theta_1) * (P_2 - P_1) * PA^{K-1} / P_0^K * (V_2 - V_1)^2 = \\ &= (\beta_2 - \beta_1) * (P_2 - P_1) * (PA / P_0)^{K-1} / (V_2 - V_1)^2 = -Ri\end{aligned}$$

This result is contrary to that reported by F.H.Ludlam in the Quarterly Journal of the Royal Meteorological Society, 1967, volume 93, 419-435. I believe an error occurs in Ludlam's calculation, causing him to assert that $Ri < \frac{1}{4}$ causes turbulence. Gary L. Russell